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Solutions of Yang's Euclidean *R*-Gauge Equations and Self-Duality

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Under some assumptions and transformations of variables, Yang's equations for *R*-gauge fields on Euclidean space lead to conformally invariant equations permitting one to obtain infinitely many other solutions from any solution of these conformally invariant equations. These conformally invariant equations closely resemble the mathematically interesting generalized Lund–Regge equations. Some exact solutions of these conformally in variant equations are obtained. Except for some singular situations, these solutions are self-dual.

1. INTRODUCTION

While discussing the self-dual $SU(2)$ gauge-fields on Euclidean space Yang arrived at the following equations:

$$\phi(\phi_{y\bar{y}} + \phi_{z\bar{z}}) - \phi_y\phi_{\bar{y}} - \phi_z\phi_{\bar{z}} + \rho_y\bar{\rho}_{\bar{y}} + \rho_z\bar{\rho}_{\bar{z}} = 0 \tag{1.1a}$$

$$\phi(\rho_{y\bar{y}} + \rho_{z\bar{z}}) - 2\rho_y\phi_{\bar{y}} - 2\rho_z\phi_{\bar{z}} = 0 \tag{1.1b}$$

where an overbar denotes the complex conjugate, ϕ and ρ are functions of y, $\bar{y}$, z, and $\bar{z}$, ϕ is real, ρ is complex, and

$$\sqrt{2}y = x^1 + ix^2 \tag{1.1c}$$

$$\sqrt{2}z = x^3 - ix^4 \tag{1.1d}$$

x^1, x^2, x^3, x^4 are real.

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In writing these equations some superfluous equations written by Yang (1977) have been dropped.

Once one has found ρ and ϕ, the corresponding R-gauge potentials are given by (Yang, 1977)

$$\phi b_y = (i\rho_y, \rho_y, -i\phi_y) \quad (1.2a)$$

$$\phi b_y = (-i\bar{\rho}_{\bar{y}}, \bar{\rho}_{\bar{y}}, i\phi_y) \quad (1.2b)$$

$$\phi b_z = (i\rho_z, \rho_z, -i\phi_z) \quad (1.2c)$$

$$\phi b_{\bar{z}} = (-i\bar{\rho}_{\bar{z}}, \bar{\rho}_{\bar{z}}, i\phi_z) \quad (1.2d)$$

and R-gauge field strengths $F_{\mu\nu}$ are given by (Yang, 1977)

$$F_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu} - B_\mu B_\nu + B_\nu B_\mu \quad (1.3a)$$

$$B_\mu = b^i_\mu x_i \quad (1.3b)$$

$$x_i = -{}^1\!/_2 i\sigma_i \quad (1.3c)$$

where σ_i are 2×2 Pauli matrices.

All such solutions satisfy the condition of self-duality except when ϕ is zero. When ϕ is zero, $F_{\mu\nu}$ becomes singular and the solutions obtained can only be treated as solutions of Yang's R-gauge equations and not self-dual solutions, unless a transformation like $F'_{\mu\nu} \to U^{-1}F_{\mu\nu}U$ removes the singularities. (These solutions may have some yet unknown relevance in the future.)

It may be noted that all self-dual solutions are known in a different form (Chirst and Weinberg, 1978). Still, a way to obtain self-dual solutions using Yang's formalism has added interest because of the simplicity and the straightforwardness of Yang's formalism. The set of equations (1.1) are important from the mathematical point of view, too. It has been observed by Jimbo *et al.* (1982) that the set of equations pass the Painlevé test for integrability in the sense of Weiss *et al.* (1983). In this present paper we show that under some assumptions and transformations equations (1.1a) and (1.1b) reduce to a conformally invariant set of equations which are similar in form to the generalized Lund–Rugge (Corones, 1978; Ray, 1982) equations. The advantage one gets from this observation is that from any solution of this reduced form one can generate infinitely many other solutions.

Some particular solutions of equations (1.1a) and (1.1b) were given by Yang himself. These solutions were generalized by Ray (1980). Two separate classes of solutions were given jointly by De and Ray (1981) in a subsequent paper. Chanda and Ray (1985) generalized the solutions obtained by Yang in a different way. These solutions included some particular cases of their generalizations reported by Ray (1980) and De and Ray (1981) as well. This paper presents infinitely many other solutions.

2. FORMULATION

When written in terms of real variables, equations (1.1a) and (1.1b) read

$$\phi(\phi_{11} + \phi_{22} + \phi_{33} + \phi_{44}) - (\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) + [(\alpha_1 + \beta_2)^2 + (\alpha_2 - \beta_1)^2 + (\alpha_3 - \beta_4)^2 + (\alpha_4 + \beta_3)^2] = 0 \tag{2.1a}$$

$$\phi(\alpha_{11} + \alpha_{22} + \alpha_{33} + \alpha_{44}) - 2[(\alpha_1 + \beta_2)\phi_1 + (\alpha_2 - \beta_1)\phi_2 + (\alpha_3 - \beta_4)\phi_3 + (\alpha_4 + \beta_3)\phi_4] = 0 \tag{2.1b}$$

$$\phi(\beta_{11} + \beta_{22} + \beta_{33} + \beta_{44}) - 2[(\beta_1 - \alpha_2)\phi_1 + (\alpha_1 + \beta_2)\phi_2 + (\alpha_4 + \beta_3)\phi_3 + (\beta_4 - \alpha_3)\phi_4] = 0 \tag{2.1c}$$

where

$$\rho = \alpha + i\beta \tag{2.1d}$$

The solutions for (2.1) presented here are those which satisfy the relations

$$\alpha = \alpha(\tau, \sigma) \tag{2.2a}$$

$$\beta = \beta(\tau, \sigma) \tag{2.2b}$$

$$\phi = \phi(\tau, \sigma) \tag{2.2c}$$

$$\tau = \tau(x^1, x^2) \tag{2.2d}$$

$$\sigma = \sigma(x^3, x^4) \tag{2.2e}$$

The solutions for (2.1) subject to (2.2) are given by the solutions of the equations (Appendix A)

$$(\phi\phi_{\tau\tau} - \phi_\tau^2 + \alpha_\tau^2 + \beta_\tau^2 + P\phi\phi_\tau)\psi + (\phi\phi_{\sigma\sigma} - \phi_\sigma^2 + \alpha_\sigma^2 + \beta_\sigma^2 + Q\phi\phi_\sigma)\chi = 0 \tag{2.3a}$$

$$(\phi\alpha_{\tau\tau} - 2\alpha_\tau\phi_\tau + P\phi\alpha_\tau)\psi + (\phi\alpha_{\sigma\sigma} - 2\alpha_\sigma\phi_\sigma + Q\phi\alpha_\sigma)\chi = 0 \tag{2.3b}$$

$$(Q\beta_{\tau\tau} - 2\beta_\tau\phi_\tau + P\phi\beta_\tau)\psi + (\phi\beta_{\sigma\sigma} - 2\beta_\sigma\phi_\sigma + Q\phi\beta_\sigma)\chi = 0 \tag{2.3c}$$

where

$$(\tau_{11} + \tau_{22})/(\tau_1^2 + \tau_2^2) = P(\tau) \tag{2.3d}$$

$$(\tau_1^2 + \tau_2^2) = \psi(\tau) \tag{2.3e}$$

$$(\sigma_{33} + \sigma_{44})/(\sigma_3^2 + \sigma_4^2) = Q(\sigma) \tag{2.3f}$$

$$(\sigma_3^2 + \sigma_4^2) = \chi(\sigma) \tag{2.3g}$$

Equations (2.3d)–(2.3g) can be rewritten as

$$\nu_{11} + \nu_{22} = 0 \tag{2.4a}$$

$$\nu_1^2 + \nu_2^2 = R \tag{2.4b}$$

$$\delta_{33} + \delta_{44} = 0 \tag{2.4c}$$

$$\delta_3^2 + \delta_4^2 = S \tag{2.4d}$$

where

$$\nu = \int \left\{ \exp\left[-\int P(\tau)\, d\tau \right] \right\} d\tau \tag{2.4e}$$

$$\delta = \int \left\{ \exp\left[-\int Q(\sigma)\, d\sigma \right] \right\} d\sigma \tag{2.4f}$$

$$R = \left\{ \exp\left[-2 \int P(\tau)\, d\tau \right] \right\} \psi(\tau) \tag{2.4g}$$

$$S = \left\{ \exp\left[-2 \int Q(\sigma)\, d\sigma \right] \right\} \chi(\sigma) \tag{2.4h}$$

By virtue of (2.4e) and (2.4f) one can consider R and S as functions of ν and δ, respectively.

The solutions for (2.4a) and (2.4b) are given by (Appendix B)

(i) $$\nu = K_2 x^1 + K_3 x^2 + K_1 \tag{2.5a}$$

$$R = K_2^2 + K_3^2 \tag{2.5b}$$

or

(ii) $$\nu = (1/2K_4) \ln[(K_4 x^1 + K_5)^2 + (K_4 x^2 + K_6)^2] + (\ln K_7)/(2K_4) \tag{2.6a}$$

$$R = 1/[(K_4 x^1 + K_5)^2 + (K_4 x^2 + K_6)^2] \tag{2.6b}$$

The solutions for (2.4c) and (2.4d) are given by (Appendix B)

$$\text{(i)} \quad \delta = K_9 x^3 + K_{10} x^4 + K_8 \tag{2.7a}$$

$$S = K_9^2 + K_{10}^2 \tag{2.7b}$$

or

$$\text{(ii)} \quad \delta = (1/2K_{11}) \ln[(K_{11}x^3 + K_{12})^2 + (K_{11}x^4 + K_{13})^2] + (\ln K_4)/(2K_{11}) \tag{2.8a}$$

$$S = 1/[(K_{11}x^3 + K_{12})^2 + (K_{11}x^4 + K_{13})^2] \tag{2.8b}$$

Now, without any loss of generality one can transform (τ, σ) to (ν, δ), and equations (2.3a)–(2.3c) lead to

$$(\phi\phi_{\nu\nu} - \phi_\nu^2 + \alpha_\nu^2 + \beta_\nu^2)R + (\phi\phi_{\delta\delta} - \phi_\delta^2 + \alpha_\delta^2 + \beta_\delta^2)S = 0 \tag{2.9a}$$

$$(\phi\alpha_{\nu\nu} - 2\alpha_\nu\phi_\nu)R + (\phi\alpha_{\delta\delta} - 2\alpha_\delta\phi_\delta)S = 0 \tag{2.9b}$$

$$(\phi\beta_{\nu\nu} - 2\beta_\nu\phi_\nu)R + (\phi\beta_{\delta\delta} - 2\beta_\delta\phi_\delta)S = 0 \tag{2.9c}$$

where ν, δ, R, and S are given by (2.5)–(2.8).

Regarding equations (2.3) and (2.9), the following observations are of interest.

I. If τ and σ satisfy a set of coupled equations of the form (2.3), then any function of τ and σ also satisfies the set of coupled equations of the form (2.3).

II. For all possible equations of the form (2.3) generated as a result of the transformation of (2.2), τ is any function of $(K_2x^1 + K_3x^2 + K_1)$ or $[(K_4x^1 + K_5)^2 + (K_4x^2 + K_6)^2]$ and σ is any function of $(K_9x^3 + K_{10}x^4 + K_8)$ or $[(K_{11}x^3 + K_{12})^2 + (K_{11}x^4 + K_{13})^2]$. Moreover, all such transformed equations are equivalent to the set of equations (2.9) via (2.4).

We consider some examples.

(i) Equations (2.3) along with

$$\tau = \ln(K_2x^1 + K_3x^2 + K_1) \tag{2.10a}$$

$$\sigma = \ln(K_9x^3 + K_{10}x^4 + K_8) \tag{2.10b}$$

or

$$\tau = (K_2x^1 + K_3x^2 + K_1)^2 \tag{2.11a}$$

$$\sigma = (K_9x^3 + K_{10}x^4 + K_8)^2, \quad \text{etc.} \tag{2.11b}$$

are equivalent to equations (2.9) along with (2.5) and (2.7)

(ii) Equations (2.3) along with

$$\tau = (K_4x^1 + K_5)^2 + (K_4x^2 + K_6)^2 \tag{2.12a}$$

$$\sigma = (K_{11}x^3 + K_{12})^2 + (K_{11}x^4 + K_{13})^2 \tag{2.12b}$$

or

$$\tau = \exp[(K_4x^1 + K_5)^2 + (K_4x^2 + K_6)^2] \qquad (2.13a)$$

$$\sigma = \exp[(K_{11}x^3 + K_{12})^2 + (K_{11}x^4 + K_{13})^2], \quad \text{etc.} \qquad (2.13b)$$

are equivalent to equation (2.9) along with (2.6) and (2.8)

(iii) Equations (2.3) along with

$$\tau = \ln(K_2x^1 + K_3x^2 + K_1) \qquad (2.14a)$$

$$\sigma = (K_{11}x^3 + K_{12})^2 + (K_{11}x^4 + K_{13})^2 \qquad (2.14b)$$

or

$$\tau = (K_2x^1 + K_3x^2 + K_1)^2 \qquad (2.15a)$$

$$\sigma = \exp[(K_{11}x^3 + K_{12})^2 + (K_{11}x^4 + K_{13})^2], \quad \text{etc.} \qquad (2.15b)$$

are equivalent to equations (2.3) along with (2.5) and (2.8)

(iv) Equations (2.3) along with,

$$\tau = (K_4x^1 + K_5)^2 + (K_4x^2 + K_6)^2 \qquad (2.16a)$$

$$\sigma = \ln(K_9x^3 + K_{10}x^4 + K_8) \qquad (2.16b)$$

or

$$\tau = \exp[(K_4x^1 + K_5)^2 + (K_4x^2 + K_6)^2] \qquad (2.17a)$$

$$\sigma = (K_9x^3 + K_{10}x^4 + K_8)^2, \quad \text{etc.} \qquad (2.17b)$$

are equivalent to equations (2.3) along with (2.6) and (2.7). Equations (2.9) reduce to an interesting form when R = const and S = const. After a transformation $(\nu, \delta) \to (\nu', \delta')$, where $\nu' = \nu/\sqrt{R}$ and $\delta' = \delta/\sqrt{S}$, one gets from (2.9)

$$(\phi\phi_{\nu'\nu'} - \phi^2_{\nu'} + \alpha^2_{\nu'} + \beta^2_{\nu'}) + (\phi\phi_{\delta'\delta'} - \phi^2_{\delta'} + \alpha^2_{\delta'} + \beta^2_{\delta'}) = 0 \qquad (2.18a)$$

$$(\phi\alpha_{\nu'\nu'} - 2\alpha_{\nu'}\phi_{\nu'}) + (\phi\alpha_{\delta'\delta'} - 2\alpha_{\delta'}\phi_{\delta'}) = 0 \qquad (2.18b)$$

$$(\phi\beta_{\nu'\nu'} - 2\beta_{\nu'}\phi_{\nu'}) + (\phi\beta_{\delta'\delta'} - 2\beta_{\delta'}\phi_{\delta'}) = 0 \qquad (2.18c)$$

Finally, (2.18) can be rewritten as

$$\Phi_{\nu'\nu'} + \Phi_{\delta'\delta'} + [(\alpha^2_{\nu'} + \beta^2_{\nu'}) + (\alpha^2_{\delta'} + \beta^2_{\delta'})] \exp(-2\Phi) = 0 \qquad (2.19a)$$

$$[\alpha_{\nu'} \exp(-2\Phi)]_{\nu'} + [\alpha_{\delta'} \exp(-2\Phi)]_{\delta'} = 0 \qquad (2.19b)$$

$$[\beta_{\nu'} \exp(-2\Phi)]_{\nu'} + [\beta_{\delta'} \exp(-2\Phi)]_{\delta'} = 0 \qquad (2.19c)$$

where

$$\phi = \exp(\Phi) \tag{2.19d}$$

The set of equations (2.19) is conformally invariant, i.e., the form of these equations is retained under any transformation $(\nu', \delta') \to (f, q)$, where f and q are functions of ν' and δ' such that $f_{\nu'} = q_{\delta'}$ and $f_{\delta'} = -q_{\nu'}$, i.e., f and q are mutually conjugate solutions of the Laplace equation in ν' and δ'. Hence from any solution of the equations (2.19) one can immediately generate infinitely many other solutions of equations (2.19) simply to replacing (ν', δ') by (f, q).

In this context it may be worthwhile to consider another set of coupled equations, namely the generalized Lund–Regge equations:

$$\theta_{11} + \theta_{22} - 4g(\theta) + h(\theta)[\lambda_1^2 \pm \lambda_2^2] = 0 \tag{2.20a}$$

$$\left[\lambda_1 \exp\left\{-\int p(\theta)\, d\theta\right\}\right]_1 + \left[\lambda_2 \exp\left\{-\int p(\theta)\, d\theta\right\}\right]_2 = 0 \tag{2.20b}$$

where $\theta = \theta(x^1, x^2)$, $\lambda = \lambda(x^1, x^2)$, $\theta_1 = \partial\theta/\partial x^1$, and so on.

With $g = 0$, equations (2.20) reduce to a conformally invariant set of equations, a particular example of which is the physically interesting equations of two-dimensional Heisenberg ferromagnets. The set of equations (2.9) closely resembles this situation, with, however, at least the difference that there are two equations for the Heisenberg ferromagnets, whereas (2.20) consists of three equations.

3. SOLUTIONS

In this paper we present some exact solutions of (2.19) which are considerably general in nature. Four interesting cases have been observed.

Case I: $\alpha = \alpha(\phi)$, $\beta = \beta(\phi)$, which can be identified with the work of De and Ray (1981).

Case II: $\alpha = \alpha(\beta)$ when the set of three equations (2.19) reduces to a set of two equations similar to the set of two equations of two-dimensional Heisenberg ferromagnets and can be solved using the procedure of Trimper (1979) and Ray (1980).

Case III: Here

$$\alpha = K_{15}\beta + u(\Phi) \tag{3.1}$$

where K_{15} is an arbitrary constant and $u(\Phi)$ is an unspecified function of Φ, $u(\Phi) \neq 0$.

Using (3.1) in (2.19b) and then using (2.18c) in the resulting expression, one gets

$$\{u_{\nu'} \exp(-2\Phi)\}_{\nu'} + \{u_{\delta'} \exp(-2\Phi)\}_{\delta'} = 0 \tag{3.2}$$

Defining

$$X = \int \exp(-2\Phi)\, du \tag{3.3a}$$

one can reduce (3.2) to

$$X_{\nu'\nu'} + X_{\delta'\delta'} = 0 \tag{3.3b}$$

which is the Laplace equation and standard solutions for X in terms of ν' and δ' are obtainable. With (3.1) and (3.3) equation (2.19b) now becomes equivalent to (2.19c).

Since the set of equations (2.19) is conformally invariant, the transformation $(\nu', \delta') \to (X, Y)$, where X and Y are mutually conjugate solutions of the Laplace equations, keeps the form of equations (2.19) unchanged.

But now, from (3.3),

$$\Phi = \Phi(X), \qquad u = u(X) \tag{3.4}$$

Thus, using the transformation $(\nu', \delta') \to (X, Y)$, (3.1), (3.3a), and (3.4), one can observe that the three equations in (2.19) reduce to two equations only, and after some rearrangement can be written as

$$\beta_X^2 + \beta_Y^2 + [(2K_{15})/(K_{15}^2 + 1)]\beta_X \exp(2\Phi)$$
$$= -[\Phi_{XX} \exp(2\Phi) + \exp(4\Phi)]/(K_{15}^2 + 1) \tag{3.5a}$$

and

$$\beta_{XX} + \beta_{YY} - 2\beta_X\Phi_X = 0 \tag{3.5b}$$

respectively.

Defining

$$\beta = \Theta - K_{15}\left[\int \exp(2\Phi)\, dX\right]/(K_{15}^2 + 1) \tag{3.6}$$

one can reduce (3.5) to

$$\Theta_X^2 + \Theta_Y^2 = M(X) \tag{3.7a}$$

where

$$M(X) = -[(K_{15}^2 + 1)\Phi_{XX} \exp(2\Phi) + \exp(4\Phi)]/(K_{15}^2 + 1)^2 \tag{3.7b}$$

and

$$\Theta_{XX} + \Theta_{YY} - 2\Theta_X\Phi_X = 0 \tag{3.7c}$$

From (3.7a), one can examine four cases separately. However, the three cases

(i) $\Theta_X = \Theta_Y = 0$

(ii) $\Theta_X \neq 0$ and $\Theta_Y = 0$

(iii) $\Theta_X = 0$ and $\Theta_Y \neq 0$

can be grouped under Θ_Y = const, where the constant may even take the value zero. However, $\Theta_Y = 0$ represents $\beta = \beta(\phi)$ and hence $\alpha = \alpha(\phi)$ from (3.1), which was considered by De and Ray (1981).

To study the fourth case, i.e., $\Theta_X \neq 0$, $\Theta_Y \neq 0$, one can proceed as follows. Differentiating (3.7a) with respect to Y, one gets

$$\Theta_X\Theta_{XY} + \Theta_Y\Theta_{YY} = 0$$

with the help of which Θ_{YY} can be eliminated from (3.7b). Doing some manipulation in the resulting expression and then on integration once, one gets

$$(\Theta_X/\Theta_Y)\exp(-2\Phi) = \pi(Y) \tag{3.8}$$

where $\pi(Y)$ is an unspecified function of Y.

This readily gives

$$\Theta = \Theta(w)$$

where

$$w = u + v$$

$$u = \int \exp(2\Phi)\, dX \qquad \text{[from (3.3a)]}$$

$$v = \int dY/\pi$$

with the use of which in (3.7a) one gets

$$(u_X^2/M) + (v_Y^2/M) = 1/\Theta_w^2 \tag{3.9}$$

Differentiating (3.9) separately with respect to u and v, respectively and comparing the results, one gets

$$M(u_X^2/M)_u + M(1/M)_u v_Y^2 = (v_Y^2)_v \tag{3.10}$$

Differentiating (3.10) successively with respect to u and v, respectively, one finally gets

$$[M(1/M)_u]_u(v_Y^2)_v = 0$$

Hence

$$M(1/M)_u = \text{const} \tag{3.11a}$$

or

$$v_Y = \text{const} \tag{3.11b}$$

That (3.11a) is not permitted in our basic assumption of (3.3a) is shown in Appendix C.

In the following we consider $v_Y = \text{const}$.

Differentiating (3.9) with respect to v and using $v_Y = \text{const}$, one gets $(1/\Theta_w^2)_w = 0$, which gives $\Theta_w = \text{const}$. Hence, $\Theta_Y = \Theta_w w_Y = \Theta_w v_Y = \text{const}$.

Thus, in this case of $\Theta_X \neq 0$, $\Theta_Y \neq 0$, too, $\Theta_Y = \text{const}$ is satisfied.

So, to find the solution for the general case, when

$$\Theta_Y = \text{const} = K_{16} \quad \text{(say)} \tag{3.12}$$

one can proceed as follows.

Using (3.12) in (3.7b) and then integrating, once one gets

$$\Theta_X = K_{17} \exp(2\Phi) \tag{3.13}$$

where K_{17} is an arbitrary constant of integration.

Generalizing (3.12) and (3.13), one concludes that

$$\Theta = K_{17} \int \exp(2\Phi)\, dX + K_{16} Y + K_{18} \tag{3.14}$$

With the use of (3.6), (3.14) reduces to

$$\beta = K_{19} \int \exp(2\Phi)\, dX + K_{16} Y + K_{18} \tag{3.15}$$

where $K_{19} = K_{17} - K_{15}/(K_{15}^2 + 1)$

It may be noted from (3.1), (3.4), and (3.15) that both α and β become functions of Φ only, when $K_{16} = 0$. This case has been treated by De and Ray (1981).

Using (3.15) in (3.5a) and rearranging, one gets

$$\begin{aligned} \Phi_{XX} = & -[K_{19}^2(K_{15}^2 + 1) + 2K_{19}K_{15} + 1] \exp(2\Phi) \\ & - K_{16}^2(K_{15}^2 + 1)\exp(-2\Phi) \end{aligned} \tag{3.16}$$

On integration, (3.15) and (3.16) lead to

$$\phi = K_{25} \operatorname{Cn} r \tag{3.17a}$$

$$\begin{aligned} \beta = & K_{19}(K_{22}^2 + K_{19}^2)^{-1/2}(K_{25}^2 + K_{24}^2)^{-1/2}[(K_{25}^2 + K_{24}^2)E(r) \\ & - K_{24}^2 r] + K_{16} Y + K_{18} \end{aligned} \tag{3.17b}$$

where

$$r = (K_{22}^2 + K_{19}^2)^{1/2}(K_{25}^2 + K_{24}^2)^{1/2}(X - K_{21}) \tag{3.17c}$$

$$K_{22} = K_{19}K_{15} + 1 \tag{3.17d}$$

$$K_{23} = K_{16}K_{15} \tag{3.17e}$$

where $K_{16} \neq 0$.

K_{20} and K_{21} are arbitrary constants of integration.

Here,

$$K_{24}^2 = \{-K_{20} - [K_{20}^2 + 4(K_{23}^2 + K_{16}^2)(K_{22}^2 + K_{19}^2)]^{1/2}\}/2(K_{22}^2 + K_{19}^2) \tag{3.17f}$$

$$K_{25}^2 = \{K_{20} - [K_{20}^2 + 4(K_{23}^2 + K_{16}^2)(K_{22}^2 + K_{16}^2)]^{1/2}\}/2(K_{22}^2 + K_{19}^2) \tag{3.17g}$$

The requirement that the permitted values of ϕ lie between $+K_{25}$ and $-K_{25}$ enables one to avoid the possible singularities.

To find the value of α, one may use the value of β from equation (3.17b) and the value of $u(\Phi)$ from equation (3.3a) and obtain

$$\alpha = K_{19}(K_{22}^2 + K_{19}^2)^{-1/2}(K_{25}^2 + K_{24}^2)^{-1/2}(1 + K_{15})[(K_{25}^2 + K_{24}^2)E(r) - K_{24}^2 r] + K_{15}K_{16}Y + K_{15}K_{18} \tag{3.17h}$$

Case IV: Without loss of generality one can write from equation (2.18b)

$$\alpha_{\nu'} \exp(-2\Phi) = \zeta_{\delta'} \tag{3.18a}$$

$$\alpha_{\delta'} \exp(-2\Phi) = -\zeta_{\nu'} \tag{3.18b}$$

such that $\alpha_{\nu'\delta'} = \alpha_{\delta'\nu'}$ leads to

$$\{\zeta_{\nu'} \exp(2\Phi)\}_{\nu'} + \{\zeta_{\delta'} \exp(2\Phi)\}_{\delta'} = 0 \tag{3.19}$$

Similarly, one can write without loss of generality from equation (2.18c),

$$\beta_{\nu'} \exp(-2\Phi) = \Sigma_{\delta'} \tag{3.20a}$$

$$\beta_{\delta'} \exp(-2\Phi) = -\Sigma_{\nu'} \tag{3.20b}$$

such that $\beta_{\nu'\delta'} = \beta_{\delta'\nu'}$ leads to

$$\{\Sigma_{\nu'} \exp(2\Phi)\}_{\nu'} + \{\Sigma_{\delta'} \exp(2\Phi)\}_{\delta'} = 0 \tag{3.21}$$

Eliminating $\alpha_{\nu'}$, $\alpha_{\delta'}$, $\beta_{\nu'}$, and $\beta_{\delta'}$ from equation (2.18a) with use of (3.18) and (3.20), one gets

$$\Phi_{\nu'\nu'} + \Phi_{\delta'\delta'} + [(\zeta_{\nu'}^2 + \Sigma_{\nu'}^2) + (\zeta_{\delta'}^2 + \Sigma_{\delta'}^2)] \exp(2\Phi) = 0 \tag{3.22}$$

In the following, we will obtain solutions of the three coupled equations (3.19), (3.21), and (3.22) using the assumption,

$$\zeta = K_{25}\Sigma + m(\Phi) \tag{3.23}$$

where K_{25} is an arbitrary real constant and $m(\Phi)$ is an unspecified function of Φ = const.

Using (3.23) in equation (3.19) and then using (3.21) in the resulting expression, one gets

$$\{m_{\nu'} \exp(2\Phi)\}_{\nu'} + \{m_{\delta'} \exp(2\Phi)\}_{\delta'} = 0 \tag{3.24}$$

Defining

$$\xi = \int \exp(2\Phi)\, dm \tag{3.25a}$$

one can reduce (3.24) to

$$\xi_{\nu'\nu'} + \xi_{\delta'\delta'} = 0 \tag{3.25b}$$

which is the Laplace equation and standard solutions of ξ in terms of ν and δ are obtainable.

With equations (3.23) and (3.25), equation (3.21) now becomes equivalent to (3.19).

Since the set of equations (3.19), (3.21), and (3.22) is conformally invariant, the transformation $(\nu', \delta') \rightarrow (\xi\ \eta)$, where ξ and η are mutually conjugate solutions of the Laplace equations, keeps the form of equations (3.19), (3.21), and (3.22) unchanged.

But now, from equation (3.25),

$$\Phi = \Phi(\xi), \qquad m = m(\xi) \tag{3.26}$$

Thus, using the transformation $(\nu', \delta') \rightarrow (\xi, \eta)$ along with equations (3.23), (3.25a), and (3.26), one can observe that the three equations (3.19), (3.21), and (3.22) reduce to two equations only and after some rearrangement can be written as

$$\begin{aligned} &\Sigma_\xi^2 + \Sigma_\eta^2 + [2K_{25}/(K_{25}^2 + 1)]\Sigma_\xi \exp(-2\Phi) \\ &\quad = -[\Phi_{\xi\xi} \exp(-2\Phi) + \exp(-4\Phi)]/(K_{25}^2 + 1) \end{aligned} \tag{3.27a}$$

and

$$\Sigma_{\xi\xi} + \Sigma_{\eta\eta} + 2\Sigma_\xi\Phi_\xi = 0 \tag{3.27b}$$

Defining

$$\Sigma = \chi - [K_{25}/(K_{25}^2 + 1)] \int \exp(-2\Phi)\, d\xi \tag{3.27c}$$

and $\chi = \chi(\xi, \eta)$, one can rewrite (3.27) as

$$\chi_\xi^2 + \chi_\eta^2 = N(\xi) \tag{3.28a}$$

where

$$N(\xi) = -[(K_{25}^2 + 1)\Phi_{\xi\xi} \exp(-2\Phi) + \exp(-4\Phi)]/(K_{25}^2 + 1)^2 \tag{3.28b}$$

$$\chi_{\xi\xi} + \chi_{\eta\eta} + 2\chi_\xi\Phi_\xi = 0 \tag{3.28c}$$

One may observe the similarities between equations (3.5) and (3.27) or between equations (3.7) and (3.28). Thus, the procedure adopted in the case of (3.7) holds here also.

Proceeding from (3.28), similarly as was done in starting from (3.7) up to (3.12), here also one can observe that

$$\chi = \chi(m + n)$$

where $m = \int \exp(-2\Phi)\, d\xi$ and $n = \int d\eta/\epsilon$, ϵ being an unspecified function of η. As before [as in equation (3.11)], these lead to

$$n_\eta = \text{const}$$

or

$$N(1/N)_m = \text{const}$$

Similar to (3.11a), it can be shown that $N(1/N)_m = \text{const}$ is not permitted (similar to Appendix C).

In the following we will consider

$$\chi = \text{const} = K_{26} \quad \text{(say)} \tag{3.29}$$

Proceeding similarly as was done starting from (3.12) up to (3.15), here one obtains

$$\Sigma = K_{27} \int \exp(-2\Phi)\, d\xi + K_{26}\eta + K_{28} \tag{3.30}$$

Using (3.30) in (3.27a) and rearranging, one gets

$$\Phi_{\xi\xi} = -[K_{29}^2(K_{25}^2 + 1) + 2K_{25}K_{29} + 1] \exp(-2\Phi) - K_{26}^2(K_{25}^2 + 1) \exp(2\Phi) \tag{3.31}$$

Also, using (3.31), it can be shown that

$$\xi = \pm(K_{33}^2 + K_{26}^2)^{-1/2} \int [(\phi^2 + K_{34}^2)(K_{35}^2 - \phi^2)]^{-1/2}\, d\phi + K_{31} \tag{3.32}$$

Here also one can observe that the integral in (3.32) has the form of an elliptical integral and can be expressed in terms of standard elliptic integrals.

On integration, (3.30) and (3.31) lead to

$$\phi = K_{31} \operatorname{Cn} r_1 \tag{3.33a}$$

$$\Sigma = \mp K_{27}(K_{33}^2 + K_{26}^2)^{-1/2}[(K_{35}^2 + K_{34}^2)E(r_1) - K_{34}^2 r_1] + K_{26}\eta + K_{28} \tag{3.33b}$$

where

$$r_1 = \mp(K_{33}^2 + K_{26}^2)^{1/2}(K_{35}^2 + K_{34}^2)^{1/2}(\xi - K_{31}) \tag{3.33c}$$

$$K_{32} = K_{29}K_{25} + 1 \tag{3.33d}$$

$$K_{33} = K_{26}K_{25} \tag{3.33e}$$

K_{30} and K_{31} are arbitrary constants of integration. Here,

$$K_{34}^2 = \{-K_{30}[K_{30}^2 + 4(K_{32}^2 + K_{29}^2)(K_{33}^2 + K_{26}^2)]^{1/2}\}/2(K_{33}^2 + K_{26}^2) \tag{3.33f}$$

$$K_{35}^2 = \{K_{30} - [K_{30}^2 + 4(K_{32}^2 + K_{29}^2)(K_{33}^2 + K_{26}^2)]^{1/2}\}/2(K_{33}^2 + K_{26}^2) \tag{3.33g}$$

Thus ϕ is given by (3.33a). Then Σ is given by (3.33b) and m is given by (3.25a), so that ζ is given by (3.23). All these quantities are given in terms of ξ and η, which are mutually conjugate solutions of the Laplace equation (3.25b).

Since equations (3.28) and hence (3.27) have been completely solved, one can now conclude that for these solutions, $\alpha_{\nu'\delta'} = \alpha_{\delta'\nu'}$ and $\beta_{\nu'\delta'} = \beta_{\delta'\nu'}$ are satisfied.

Hence, from (3.18) one can write

$$\alpha = \int [\zeta_{\delta'} \exp(2\Phi)]\, d\nu' + \int \left\{-\zeta_{\nu'} \exp(2\Phi) - \frac{\partial}{\partial\delta'} \int [\zeta_{\delta'} \exp(2\Phi)\, d\nu']\right\} d\delta' + K_{36}$$

which with the use of (2.18d) reduces to

$$\alpha = \int (\phi^2\zeta_{\delta'})\, d\nu' + \int \left[-\phi^2\zeta_{\nu'} - \frac{\partial}{\partial\delta'} \int (\phi^2\zeta_{\delta'})\, d\nu'\right] d\delta' + K_{36} \tag{3.34}$$

where K_{36} is an arbitrary constant of integration.

Similarly, from (3.20) and (2.18b) one can write

$$\beta = \int (\phi^2 \Sigma_{\delta'})\, d\nu' + \int \left[-\phi^2 \Sigma_{\nu'} - \frac{\partial}{\partial \delta'} \int (\phi^2 \Sigma_{\delta'})\, d\nu' \right] d\delta' + K_{37} \tag{3.35}$$

where K_{37} is another arbitrary constant of integration.

We have seen that Σ, ζ, and ϕ can be expressed in terms of ξ and η, which are mutually conjugate solutions of the Laplace equation (3.25b). Hence, one can conclude that (3.34) and (3.35) give α and β, respectively, in terms of ξ and η.

4. SUMMARY

As a result of searching for solutions of (2.1) of the type (2.2) we observe the following.

1. τ is any arbitrary function of $(K_2x^1 + K_3x^2 + K_1)$ or $\{(K_4x^1 + K_5)^2 + (K_4x^2 + K_6)^2\}$. σ is any arbitrary function of $(K_9x^1 + K_{10}x^2 + K_8)$ or $\{(K_{11}x^3 + K_{12})^2 + (K_{11}x^4 + K_{13})^2\}$.

2. For any such value of τ and σ equations (2.1) can be transformed to (2.9) via (2.2) along with (2.5)–(2.8).

3. For R = const and S = const only (2.5) and (2.7) are permitted. However, for various complicated forms of τ and σ equations (2.1) can be transformed to (2.19) with (2.5) and (2.7). Furthermore, for R = const and S = const we get from (2.9) a set of equations (2.19) which is conformally invariant and is very similar in form to the generalized Lund–Regge equations (Corones, 1978; Ray, 1982). Thus from any solutions of (2.19) one can generate infinitely many other solutions by virtue of transformations of the type $(\nu', \delta') \to (f, q)$, where (ν', δ') are old independent variables, (f, q) are new independent variables, and (f, q) are functions of (ν', δ') such that $f_{\nu'} = q_{\delta'}$ and $f_{\delta'} = -q_{\nu'}$, i.e., f and q are mutually conjugate solutions of the Laplace equations in ν' and δ'.

4. The solutions of the equation (1.1) via (2.19) observed by us can be grouped under four cases. In spite of the fact that cases I and II are to some extent repetitions of previous work (De and Ray, 1981; Trimper, 1979; Ray, 1980a,b), our observation that an infinite number of new solutions can be generated from any solution of (2.19) makes mention of these cases worthwhile in our context.

Case I: Here, $\alpha = \alpha(\phi)$, $\beta = \beta(\phi)$, and the solutions are particular cases of the solutions obtained by De and Ray (1981).

Case II: Here $\alpha = \alpha(\beta)$ and the solutions are particular cases of solutions obtainable using the procedure of Trimper (1979) and Ray (1980a,b).

Case III: ϕ is obtained in equation (3.17a), where r can be found from (3.17c); α and β are obtained in equations (3.17h) and (3.17b), respectively. In these equations, $E(r)$ is an odd analytic function (Grdelyi *et al.*, 1953) of r and when r is increased by $2K$, $E(r)$ is reproduced, save for an additive constant given by $\int_0^{2K} dn^2\, r\, dr$, where, K is the complete elliptic integral of the first kind,

$$K = \int_0^{\pi/2} (1 - k^2 \sin^2\theta)^{-1/2}\, d\theta$$

$$= \tfrac{1}{2}\pi F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2) \tag{4.1}$$

when k lies in the cut plane.

The ϕ in equation (3.17a) oscillates (Grdelyi *et al.*, 1953) between $+K_{25}$ and $-K_{25}$ with a period $4K$ and has zero points congruent with

$$X = (K_{22}^2 + K_{19}^2)^{-1/2}(K_{25}^2 + K_{24}^2)^{-1/2}K + K_{21}$$

or

$$X = 3(K_{22}^2 + K_{19}^2)^{-1/2}(K_{25}^2 + K_{24}^2)^{-1/2}K + K_{21}$$

where K is defined by (4.1).

Case IV: ϕ is obtained in equation (3.33a), where r_1 is obtained in (3.33c); α and β are obtained in equations (3.34) and (3.35), respectively. Here Σ is obtained in equation (3.33b), which also gives ζ.

r_1 has the property similar to r described above for case III. As we now know α and β, we can obtain ρ using equation (2.1d). Once we have found ρ and ϕ, the corresponding R-gauge potentials and the R-gauge field strengths can be obtained from (1.2) and (1.3), respectively.

All such solutions represent the condition of self-duality except when ϕ is zero, because where ϕ is zero, $F_{\mu\nu}$ becomes singular and the solutions obtained can only be treated as solutions of Yang's R-gauge equations and not self-dual solutions unless a transformation like $F'_{\mu\nu} \to U^{-1}F_{\mu\nu}U$ removes the singularities. However, these solutions may have some (unknown) relevance in future.

APPENDIX A

From equation (2.2c), $\phi = \phi(\tau, \sigma)$, which gives

$$\phi_1 = \phi_\tau \tau_1 \qquad \text{and} \qquad \phi_{11} = \phi_{\tau\tau}\tau_1^2 + \phi_\tau \tau_{11}$$

Similarly, we have ϕ_2, ϕ_3, ϕ_4, ϕ_{22}, ϕ_{33}, and ϕ_{44}.

Again, from (2.2a) and (2.2b) we have similar equations for ϕ's. Using these in equation (2.1a), keeping in mind equations (2.2d) and (2.2e), we have

$$
\begin{aligned}
&(\phi\phi_{\tau\tau} - \phi_\tau^2 + \alpha_\tau^2 + \beta_\tau^2)(\tau_1^2 + \tau_2^2) \\
&\quad + (\phi\phi_{\sigma\sigma} - \phi_\sigma^2 + \alpha_\sigma^2 + \beta_\sigma^2)(\sigma_3^2 + \sigma_4^2) \\
&\quad + \phi\phi_\tau(\tau_{11} + \tau_{22}) + \phi\phi_\sigma(\sigma_{33} + \sigma_{44}) = 0 \qquad \text{(A1a)}
\end{aligned}
$$

Similarly, we have from equation (2.1b)

$$
\begin{aligned}
&(\phi\alpha_{\tau\tau} - 2\alpha_\tau\phi_\tau)(\tau_1^2 + \tau_2^2) + (\phi\alpha_{\sigma\sigma} - 2\alpha_\sigma\phi_\sigma)(\sigma_3^2 + \sigma_4^2) \\
&\quad + \phi\alpha_\tau(\tau_{11} + \tau_{22}) + \phi\alpha_\sigma(\sigma_{33} + \sigma_{44}) = 0 \qquad \text{(A1b)}
\end{aligned}
$$

and from equation (2.1c)

$$
\begin{aligned}
&(\phi\beta_{\tau\tau} - 2\beta_\tau\phi_\tau)(\tau_1^2 + \tau_2^2) + (\phi\beta_{\sigma\sigma} - 2\beta_\sigma\phi_\sigma)(\sigma_3^2 + \sigma_4^2) \\
&\quad + \phi\beta_\tau(\tau_{11} + \tau_{22}) + \phi\beta_\sigma(\sigma_{33} + \sigma_{44}) = 0 \qquad \text{(A1c)}
\end{aligned}
$$

Comparing the value of $(\sigma_{33} + \sigma_{44})$ from (A1a) and (A1c) and that from (A1c) and (A1b), then dividing these two equations, we have after some simplification

$$
\begin{aligned}
&(\tau_{11} + \tau_{22})/(\tau_1^2 + \tau_2^2) \\
&\quad = \text{a function of } (\phi, \phi_\tau, \phi_\sigma, \phi_{\tau\tau}, \\
&\qquad \phi_{\sigma\sigma}, \alpha_\tau, \alpha_\sigma, \alpha_{\tau\tau}, \alpha_{\sigma\sigma}, \\
&\qquad \beta_\tau, \beta_\sigma, \beta_{\tau\tau}, \beta_{\sigma\sigma})
\end{aligned}
$$

Since the left-hand side of above equation is a function of x^1 and x^2, the right-hand side will also be function of x^1 and x^2. But on the right-hand side x^1 and x^2 do not appear in explicit form, rather as a function of $\tau(x^1, x^2)$ only. Thus we may write

$$
\begin{aligned}
(\tau_{11} + \tau_{22})/(\tau_1^2 + \tau_2^2) &= \text{an arbitrary function of } \tau \\
&= P(\tau) \quad \text{(say)} \qquad \text{(A2)}
\end{aligned}
$$

By the same procedure we arrive at

$$
\begin{aligned}
(\sigma_{33} + \sigma_{44})/(\sigma_3^2 + \sigma_4^2) &= \text{an arbitrary function of } \sigma \\
&= Q(\sigma) \quad \text{(say)} \qquad \text{(A3)}
\end{aligned}
$$

Putting (A2) and (A3) in equation (A1a) and using the same argument as above we arrive at

$$
\tau_1^2 + \tau_2^2 = \psi(\tau) \qquad \text{and} \qquad \sigma_3^2 + \sigma_4^2 = \chi(\sigma)
$$

Here $\psi(\tau)$ is an arbitrary function of τ and $\chi(\sigma)$ is another arbitrary function of σ. Then equations (A1a)–(A1c) reduce to (2.3a)–(2.3c).

APPENDIX B

First we consider the equations

$$\text{(2.4a)} \quad \nu_{11} + \nu_{22} = 0$$

$$\text{(2.4b)} \quad \nu_1^2 + \nu_2^2 = R$$

Differentiating (2.4b) first with respect to x^1, we obtain ν_{11} and then with respect to x^2, we obtain ν_{22}. Putting these values of ν_{11} and ν_{22} into equation (2.4a), we have

$$R\nu_{12} = R_\nu \nu_1 \nu_2 \tag{B1}$$

From here we have two cases:

Case I: $R_\nu = 0$. Then $R = \text{const}$. Hence, ν should be a linear function of x^1 and x^2, i.e.,

$$\nu = E(x^1) + F(x^2) + K_1'$$

Finding ν_1 and ν_2 and using (2.4b), we arrive at

$$\nu = K_2 x^1 + K_3 x^2 + K_1 \tag{B2}$$

with

$$K_2^2 + K_3^2 = R \tag{B3}$$

where K_1, K_2, and K_3 are constants.

Case II. $R_\nu \neq 0$. Writing (B1) first as $R\nu_{12} = R_1\nu_2$ and then integrating with respect to x^1, we have

$$\nu_2 = VR \tag{B4a}$$

where V is an arbitrary function of x^2 only.

Now writing (B1) as $R\nu_{12} = R_2\nu_1$ and then integrating with respect to x^2, we have

$$\nu_1 = UR \tag{B4b}$$

where U is an arbitrary function of x^1 only.

Using (B4a), (B4b) in equations (2.4a), (2.4b), we have

$$U_1 + V_2 + (U^2 + V^2)R_\nu = 0 \tag{B5a}$$

$$U^2 + V^2 = 1/R \tag{B5b}$$

Using (B5b) in equation (B5a), we arrive at

$$U_1 + V_2 = -(\ln R)_\nu \tag{B6}$$

Differentiating (B5) separately with respect to x^1 and x^2, using (B4a), (B4b), and finally comparing the results, we conclude that

$$U_1 = V_2 \tag{B7}$$

which shows that left-hand side is a function of x^1 only, whereas the right-hand side is a function of x^2 only.

Hence, one can conclude from (B7) that

$$U_1 = V_2 = K_4 \tag{B8}$$

where K_4 is an arbitrary constant.

When $U_1 = K_4$, then

$$U = K_4 x^1 + K_5 \tag{B9a}$$

When $V_2 = K_4$, then

$$V = K_4 x^2 + K_6 \tag{B9b}$$

where K_5 and K_6 are arbitrary constants of integration.

Using (B9a) and (B9b) in equation (B6), we have

$$R = K_7 \exp(-2K_4\nu) \tag{B10}$$

with $K_7 \neq 0$, $K_4 \neq 0$. Putting the value of R from equation (B10) in (B4a), (B4b), we have

$$\{\exp(2K_4\nu)\}_1 = 2K_4^2 K_7 x^1 + 2K_4 K_5 K_7 \tag{B11a}$$

$$\{\exp(2K_4\nu)\}_2 = 2K_4^2 K_7 x^2 + 2K_4 K_6 K_7 \tag{B11b}$$

Integrating (B11a), one gets

$$\exp(2K_4\nu) = K_4^2 K_7 (x^1)^2 + 2K_4 K_5 K_7 x^1 + G(x^2) \tag{B12a}$$

where $G(x^2)$ is any arbitrary function of x^2.

Using equation (B12a) in (B11b), we have

$$[G(x^2)]_2 = 2K_4^2 K_7 x^2 + 2K_4 K_6 K_7 \tag{B12b}$$

Integrating (B12b) with respect to x^2 and then using the result in (B12a), one finally gets

$$\nu = [1/(2K_4)] \ln\{K_4^2 K_7[(x^1)^2 + (x^2)^2] + 2K_4 K_7(K_5 x^1 + K_6 x^2) + K_8'\} \tag{B13}$$

Using (B13) in equation (B10), we have

$$R = K_7/\{K_4^2 K_7[(x^1)^2 + (x^2)^2] + 2K_4 K_7[K_5 x^1 + K_6 x^2] + K_8'\} \tag{B14}$$

where K_8' is an arbitrary constant of integration.

Differentiating (B13) first with respect to x^1, then with respect to x^2, and using the results along with (B14) in (2.4b), one gets

$$K_8' = K_7(K_5^2 + K_6^2) \quad \text{(B15)}$$

Using (B15) in (B13) and (B14), we have

$$\nu = [1/(2K_4)] \ln[(K_4x^1 + K_5)^2 + (K_4x^2 + K_6)^2] + [\ln K_7/(2K_4)] \quad \text{(B16)}$$

$$R = 1/[(K_4x^1 + K_5)^2 + (K_4x^2 + K_6)^2] \quad \text{(B17)}$$

One can check then (B16) identically satisfies (2.4a).

Similarly, using equations (2.4c) and (2.4d) we have: (i) For $S_\delta = 0$

$$\delta = K_9x^3 + K_{10}x^4 + K_8 \quad \text{(B18)}$$

$$S = K_9^2 + K_{10}^2 \quad \text{(B19)}$$

(ii) For $S_\delta \neq 0$

$$\delta = [1/(2K_{11})] \ln[(K_{11}x^3 + K_{12})^2 + (K_{11}x^4 + K_{13})^2] + [1/(2K_{11})] \ln K_{14} \quad \text{(B20)}$$

$$S = 1/[(K_{11}x^3 + K_{12})^2 + (K_{11}x^4 + K_{13})^2] \quad \text{(B21)}$$

where K_8, K_9, K_{10}, K_{11}, K_{12}, K_{13}, and K_{14} are arbitrary constants.

APPENDIX C

Consider

$$M(1/M)_u = \text{const} = L \quad \text{(say)} \quad \text{(C1)}$$

It is evident from equation (3.10) that

$$M(u_X^2/M)_u + M(1/M)_u v_Y^2 = (v_Y^2)_v$$

$$M(u_X^2/M) = \text{const} = C \quad \text{(say)} \quad \text{(C2)}$$

Using (C1) and (C2) in equation (3.10), it reduces to

$$(v_Y^2)_v = Lv_Y^2 + C \quad \text{(C3)}$$

From (C1) we get

$$M = N_1 \exp(-Lu) \quad \text{(C4)}$$

where $N_1 = \text{const} \neq 0$. In the following it will be shown that the above equations are not satisfied simultaneously and hence $M(1/M)_u$ = const is not possible.

Case I: $L \neq 0$. Expanding (C2), we have

$$M\,(u_X^2)_u/M + u_X^2 M(1/M)_u = C$$

Using (C1) and then integrating, we have from the above equation

$$C_1 \exp(-Lu) = C - Lu_X^2 \tag{C5}$$

where C_1 is an integration constant. Rearranging equation (C5), we have

$$u_X^2 = C_2 - C_3 \exp(-Lu) \tag{C6}$$

where $C_2 = C/L$ and $C_3 = C_1/L$. From equation (3.3a) we have

$$u_X = \exp(2\Phi) \tag{C7}$$

from which (C6) becomes

$$\exp(-Lu) = C_4 - C_5 \exp(4\Phi) \tag{C8}$$

where $C_4 = C_2/C_3$ and $C_5 = 1/C_3$ are constants.

Putting the value of M from equation (C4) into equation (3.7b) and using (C8), we have

$$\Phi_{XX} = C_6 \exp(2\Phi) + C_7 \exp(-2\Phi) \tag{C9}$$

where

$$C_6 = [N_1 C_5 (K_{15}^2 + 1)^2 - 1]/(K_{15}^2 + 1) = \text{const}$$

$$C_7 = -[N(K_{15}^2 + 1)^2 C_4] = \text{const}$$

Using (C9) in equation (3.7b), we obtain

$$M = C_8 \exp(4\Phi) + C_9 \tag{C10}$$

where

$$C_8 = -[C_6(K_{15}^2 + 1) + 1]/(K_{15}^2 + 1)^2 = \text{const}$$

$$C_9 = C_7/(K_{15}^2 + 1)$$

Comparing equations (C10) and (C4), we have

$$N_1 \exp(-Lu) = C_8 \exp(4\Phi) + C_9 \tag{C11}$$

Using (C8), (C11), and $\exp(\Phi) = \phi$, we have, on simplification,

$$\phi^4 = (C_9 - N_1 C_2/C_3)/(C_8 + N_1/C_3) \tag{C12}$$

As C_2, C_3, C_8, C_9, and N_1 are constants, (C12) leads to the trivial solution

$$\phi = \text{const} \tag{C13}$$

This proves that (C1) is not possible.

Case II: $L = 0$. Then, from equation (C4),

$$M(X) = \text{const} \tag{C14}$$

which leads to $X = \text{const}$. Hence (C1) is not possible.

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